# Nonlinear continuous-time New Keynesian models

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# Introduction

- This note provides algebraic derivations of nonlinear continuous-time New Keynesian models implemented in Continuo.
- The three flavours differ in how habits enter the household's problem and the resulting marginal cost expressions.
- All models feature a zero lower bound on the nominal interest rate, Rotemberg price adjustment costs, and a temporary productivity boom as the exogenous shock.
- Code implementing these models is available in the `examples/nk-nonlinear/` directory of the Continuo repository.

# Plan

# Common setup – Overview

- Three flavours of the fully nonlinear continuous-time New Keynesian model, all in the file `examples/nk-nonlinear/`:
  - `baseline.mod`: standard NK with no habits.
  - `external_habit.mod`: additive habits taken as given by the household (catching-up-with-the-Joneses).
  - `internal_habit.mod`: the same habits internalised, with a costate on the habit stock.
- Household preferences, the firms' Rotemberg pricing problem, the monetary-policy rule with ZLB and the TFP shock are common to all three.
- The variants differ only in how the habit stock $X$ enters the household's problem.

# Common setup – Households

- Representative household maximises

$$U \;=\; \int_0^\infty e^{-\rho t}\left[u\big(C(t),X(t)\big) - \frac{N(t)^{1+\eta}}{1+\eta}\right]dt,$$

with discount rate $\rho > 0$ and inverse Frisch elasticity $\eta$.

- Felicity $u(C,X)$ depends on a habit stock $X$ (specialised below; $h = 0$ removes it). Labour disutility is separable.

- Budget constraint, real one-period bonds $B$:

$$\dot{B}(t) \;=\; \big(R(t) - \pi(t)\big)\,B(t) + w(t)\,N(t) - C(t),$$

with $R$ the nominal policy rate, $\pi$ inflation, $w$ the real wage.

# Common setup – Firms: demand and technology

- A continuum of monopolistically competitive intermediate firms $i \in [0,1]$ supplies varieties of a single composite good. Each faces the constant-elasticity demand schedule
$$y_i = \left(\frac{p_i}{P}\right)^{-\varepsilon} Y,$$
with $P$ the aggregate price index, $Y$ aggregate output, and $\varepsilon > 1$ the elasticity of substitution between varieties.

- Linear technology $y_i = A\, n_i$ (with $A = 1$), the real marginal cost is the same across firms $MC = w/A$ .

- **Rotemberg (1982) price-adjustment cost.** Adjusting $p_i$ at rate $\pi_i = \dot{p}_i / p_i$ costs $\phi/2 \cdot \pi_i^2$ per unit of output, i.e. a real flow cost of $\frac{\phi}{2}\, \pi_i^2\, Y$.

- The target for the real marginal cost is the inverse price markup, $MC^\star = (\varepsilon - 1)/\varepsilon$. The NKPC will discipline $\dot{\pi}$ by the gap $MC - MC^\star$.

# Common setup – Firms: optimisation

- Working with the relative price $\xi_i = p_i/P$, its law of motion is
$$\dot{\xi}_i \;=\; \xi_i\,(\pi_i - \pi), \qquad \pi = \dot{P}/P.$$
- In equilibrium the firm is owned by the household, so it values future real profits at the household's stochastic discount factor
$$\Lambda(t) \;\equiv\; u'(C(t)) \;=\; C(t)^{-\sigma},$$
i.e. the marginal utility of an extra real unit of consumption.
- Firm $i$ chooses $\{\pi_i(t)\}$ to maximise
$$\int_0^\infty e^{-\rho t}\,\Lambda(t)\Big[\;\underbrace{\xi_i^{1-\varepsilon}\,Y}_{\text{real revenue}} \;-\underbrace{MC\,\xi_i^{-\varepsilon}\,Y}_{\text{real cost}}-\underbrace{\tfrac{\phi}{2}\,\pi_i^2\,Y}_{\text{adj. cost}}\Big]\,dt$$
subject to $\dot{\xi}_i = \xi_i\,(\pi_i - \pi)$, with $\xi_i(0)$ given. The CES demand $y_i = \xi_i^{-\varepsilon}Y$ is substituted into the revenue and cost terms, so it does not appear as an explicit constraint.

# Common setup – Firms: Hamiltonian

- Casting the dynamic programme above as a Hamiltonian: state $\xi_i$, control $\pi_i$, costate $\nu$ on the relative-price constraint. The current-value Hamiltonian is

$$\mathcal{H} \;=\; \Lambda\left[\xi_i^{1-\varepsilon}\,Y - MC\,\xi_i^{-\varepsilon}\,Y - \tfrac{\phi}{2}\,\pi_i^2\,Y\right] + \nu\,\xi_i\,(\pi_i - \pi).$$

- Notice the structure: the bracket is the instantaneous real profit (weighted by the household's SDF $\Lambda$), and the $\nu\,\xi_i\,(\pi_i - \pi)$ term is the costate times the state's equation of motion — the standard "flow value + multiplier × constraint" form of an optimal-control Hamiltonian.

- The next slide derives the FOC in the control and the costate equation, then takes the symmetric-equilibrium limit.

# Common setup – Firms: FOC and costate

- FOC in the control $\pi_i$:

$$\frac{\partial \mathcal{H}}{\partial \pi_i} = -\Lambda\,\phi\,\pi_i\,Y + \nu\,\xi_i = 0 \quad \Longrightarrow \quad \nu\,\xi_i \;=\; \Lambda\,\phi\,\pi_i\,Y.$$

- Costate equation $\dot{\nu} = \rho\,\nu - \partial\mathcal{H}/\partial\xi_i$, with

$$\frac{\partial \mathcal{H}}{\partial \xi_i} = \Lambda\left[(1-\varepsilon)\,\xi_i^{-\varepsilon}\,Y + \varepsilon\,MC\,\xi_i^{-\varepsilon-1}\,Y\right] + \nu\,(\pi_i - \pi).$$

- Impose symmetric equilibrium ($\xi_i = 1$, $\pi_i = \pi$, all firms identical). From the FOC, $\nu = \Lambda\,\phi\,\pi\,Y$. The last term of $\partial\mathcal{H}/\partial\xi_i$ vanishes, and the first two collapse to $\Lambda\,\varepsilon\,Y\,[MC - (\varepsilon-1)/\varepsilon]$:

$$\dot{\nu} \;=\; \rho\,\nu - \Lambda\,\varepsilon\,Y\left[MC - \tfrac{\varepsilon-1}{\varepsilon}\right].$$

- Differentiate $\nu = \Lambda\,\phi\,\pi\,Y$ exactly using the product rule:

$$\dot{\nu} \;=\; \nu\left[\frac{\dot{\Lambda}}{\Lambda} + \frac{\dot{\pi}}{\pi} + \frac{\dot{Y}}{Y}\right].$$

# Common setup – Firms: resource constraint and marginal cost

- The Rotemberg adjustment cost is a **real resource loss** paid out of aggregate output. In symmetric equilibrium,

$$Y \;=\; C + \tfrac{\phi}{2}\,\pi^2\,Y \quad\Longleftrightarrow\quad Y \;=\; \frac{C}{1-(\phi/2)\,\pi^2}.$$

- Differentiating $\log Y = \log C - \log(1-(\phi/2)\pi^2)$ yields the exact link

$$\boxed{\frac{\dot Y}{Y} \;=\; \frac{\dot C}{C} \;+\; \frac{\phi\,\pi\,\dot\pi}{1-(\phi/2)\,\pi^2}}$$

used on a later slide to eliminate $\dot Y/Y$ from the costate identity.

# Common setup – Firms: marginal cost across variants

- Production $Y = A\,N$ gives $N = Y/A$. The labour FOC $w = N^\eta / u_C$ then writes the real marginal cost $MC = w/A$ in terms of output and the model's particular marginal-utility-of- consumption expression $u_C$:

$$\begin{aligned} \text{baseline}\,(u_C = C^{-\sigma}): &\qquad MC = Y^\eta\, C^\sigma \,/\, A^{\eta+1}, \\ \text{external habit}\,(u_C = (C - hX)^{-\sigma}): &\qquad MC = Y^\eta\, (C - hX)^\sigma \,/\, A^{\eta+1}, \\ \text{internal habit}\,(u_C = \lambda_B - \lambda\mu): &\qquad MC = Y^\eta \,/\, (\lambda_B\, A^{\eta+1}). \end{aligned}$$

- At $A^\star = 1$ and $\pi = 0$, $Y^\star = C^\star$ and these collapse to the canonical no-adjustment-cost expressions used in the steady-state formulas below.

# Common setup – Firms: the exact NKPC, general $\sigma$

- Combine $\dot{\nu}/\nu = \dot{\Lambda}/\Lambda + \dot{\pi}/\pi + \dot{Y}/Y$ (slide before last) with $\dot{\Lambda}/\Lambda = -\sigma\,\dot{C}/C$ and the exact resource-constraint link $\dot{Y}/Y = \dot{C}/C + (\phi\pi\dot{\pi})/(1-(\phi/2)\pi^2)$:

$$\frac{\dot{\nu}}{\nu} \;=\; (1-\sigma)\,\frac{\dot{Y}}{Y} + \sigma\,\frac{\phi\,\pi\,\dot{\pi}}{1-(\phi/2)\,\pi^2} + \frac{\dot{\pi}}{\pi}.$$

- Equating with the costate result $\dot{\nu}/\nu = \rho - (\varepsilon/(\phi\pi))\,(MC - (\varepsilon-1)/\varepsilon)$ and multiplying by $\pi$:

$$(1-\sigma)\,\frac{\dot{Y}}{Y}\,\pi + \frac{\sigma\,\phi\,\pi^2\,\dot{\pi}}{1-(\phi/2)\,\pi^2} + \dot{\pi} \;=\; \rho\,\pi - \frac{\varepsilon}{\phi}\Big(MC - \tfrac{\varepsilon-1}{\varepsilon}\Big).$$

# Common setup – Firms: the exact NKPC, log utility

- Set $\sigma = 1$ (our calibration). The $(1-\sigma)\,\dot{Y}/Y\,\pi$ term vanishes, and the previous slide's equation reduces to

$$\dot{\pi}\left[1 + \frac{\phi\,\pi^2}{1-(\phi/2)\,\pi^2}\right] \;=\; \rho\,\pi \;-\; \frac{\varepsilon}{\phi}\left(MC - \tfrac{\varepsilon-1}{\varepsilon}\right).$$

- Using $1+\phi\pi^2/(1-(\phi/2)\pi^2) = (1+(\phi/2)\pi^2)/(1-(\phi/2)\pi^2)$ and dividing through:

$$\boxed{\dot{\pi} \;=\; \frac{1-(\phi/2)\,\pi^2}{1+(\phi/2)\,\pi^2}\left[\rho\,\pi - \tfrac{\varepsilon}{\phi}\left(MC - \tfrac{\varepsilon-1}{\varepsilon}\right)\right]}$$

- The correction factor is $1$ at $\pi = 0$.

# Common setup – Monetary policy and the TFP shock

- Taylor-type rule, clamped at the zero lower bound:

$$\boxed{R(t) \;=\; \max\bigl(0,\; \rho + \phi_\pi\, \pi(t)\bigr)}$$

  Intercept $\rho$ so that the no-shock steady state has $R^\star = \rho$, $\pi^\star = 0$. The $\max(0, \cdot)$ makes the model genuinely nonlinear.

- **TFP shock.** The only exogenous driver is total-factor productivity $A(t)$ in $Y = A\,N$.

- No-shock SS: $A^\star = 1$. The experiment is a temporary boom: $A$ rises to $1.12$ over $[0, 3)$ and returns to $1$. The boom lowers $MC$, drives deflation through the NKPC, and forces the Taylor rule onto the ZLB — the classic "supply-side deflationary trap".

# Common setup – Calibration

| symbol | value | meaning |
|---|---|---|
| $\sigma$ | $1$ | inverse IES (log utility) |
| $\eta$ | $1$ | inverse Frisch elasticity |
| $\varepsilon$ | $6$ | elasticity of substitution (20% markup) |
| $\phi$ | $40$ | Rotemberg price-adjustment cost |
| $\rho$ | $0.02$ | rate of time preference (= steady-state real rate) |
| $\phi_\pi$ | $1.5$ | Taylor-rule inflation response |
| $\lambda$ | $0.5$ | habit adjustment speed (habit variants only) |
| $h$ | $0.7$ | habit weight (habit variants only) |

# Plan

# Baseline – Euler equation

- With $h = 0$ the felicity collapses to $u(C) = C^{1-\sigma}/(1-\sigma)$, and the marginal utility of consumption is $u_C = C^{-\sigma}$.

- The continuous-time consumption Euler in a perfect-foresight setting reads $\dfrac{d \log u_C}{dt} = \rho - R + \pi$, so

$$-\sigma \frac{\dot{C}}{C} = \rho - R + \pi,$$

  which rearranges to the standard form

$$\boxed{\dot{C} \;=\; \frac{C}{\sigma}\left(R - \pi - \rho\right)}$$

- For log utility ($\sigma = 1$) this reduces to $\dot{C}/C = R - \pi - \rho$.

# Baseline – Marginal cost

- Labour FOC: $u_N = \lambda_B\, w$, with $u_N = -N^\eta$ (separable disutility) and $\lambda_B = u_C = C^{-\sigma}$ (no habit). Therefore

$$w \;=\; \frac{N^\eta}{u_C} \;=\; N^\eta\, C^\sigma.$$

- Goods-market clearing: $Y = C$ (production absorbs consumption, Rotemberg adjustment costs ignored at first order). Production gives $N = Y/A = C$ at $A = 1$, so

$$\boxed{MC \;=\; \frac{w}{A} \;=\; N^\eta\, C^\sigma \;=\; C^{\sigma+\eta}}$$

- This closes the equation system together with the Euler and the NKPC.

# Baseline – Full system

| role | equation |
|---|---|
| state | — |
| jump | $\dot{C} \;=\; (C/\sigma)\,(R-\pi-\rho)$ |
| jump | $\dot{\pi} \;=\; \frac{1-(\phi/2)\pi^2}{1+(\phi/2)\pi^2}\left[\rho\,\pi-(\varepsilon/\phi)(MC-\frac{\varepsilon-1}{\varepsilon})\right]$ |
| algebraic | $Y \;=\; C\,/\,(1-(\phi/2)\,\pi^2)$ |
| algebraic | $MC \;=\; Y^{\eta}\,C^{\sigma}/A^{\eta+1}$ |
| algebraic | $R \;=\; \max(0,\; \rho+\phi_{\pi}\,\pi)$ |

**Steady state** $(A^{\star}=1)$: $\pi^{\star}=0$ gives $Y^{\star}=C^{\star}$, hence

$$R^{\star}=\rho, \quad MC^{\star}=\frac{\varepsilon-1}{\varepsilon}, \quad C^{\star}=(MC^{\star}\cdot A^{*\,\eta+1})^{1/(\sigma+\eta)}=(MC^{\star})^{1/(\sigma+\eta)}.$$

# Plan

# External habit – Setup

- Felicity is now

$$u(C,X) = \frac{(C-hX)^{1-\sigma}}{1-\sigma}, \qquad 0 \le h < 1,$$

with a habit stock $X$ that evolves as an exponential moving average of consumption:

$$\boxed{\dot{X} = \lambda\,(C-X), \qquad \lambda > 0}$$

- **External**: the household treats $X$ as a given aggregate state — it does *not* internalise that its own $C$ raises future $X$. There is no costate on $X$ in the household's problem.

- Marginal utility is

$$u_C = (C-hX)^{-\sigma},$$

exactly as in the no-habit problem but evaluated on the habit-adjusted consumption $C-hX$.

# External habit – Euler equation

- The Euler condition $\frac{d \log u_C}{dt} = \rho - R + \pi$ becomes

$$-\sigma \frac{\dot{C} - h\,\dot{X}}{C - hX} \;=\; \rho - R + \pi.$$

- Equivalently,

$$\dot{C} - h\,\dot{X} \;=\; \frac{C - hX}{\sigma}\,(R - \pi - \rho),$$

  and substituting the habit law $\dot{X} = \lambda(C - X)$ yields the habit-augmented consumption Euler

$$\boxed{\dot{C} \;=\; \frac{C - hX}{\sigma}\,(R - \pi - \rho) + h\,\lambda\,(C - X)}$$

- When $h = 0$ this reduces to the baseline Euler $\dot{C} = (C/\sigma)(R - \pi - \rho)$.

# External habit – Marginal cost

- The labour FOC is now

$$w = \frac{N^\eta}{u_C} = N^\eta (C - hX)^\sigma.$$

- Combining with $A = 1$ and $N = C$ (goods-market clearing),

$$\boxed{MC = \frac{w}{A} = C^\eta (C - hX)^\sigma}$$

- In steady state $X^\star = C^\star$, so $C - hX = (1-h)C$ and $MC^\star = (1-h)^\sigma \, C^{*\,\sigma+\eta}$.

# External habit – Full system

| role | equation |
|---|---|
| state | $\dot{X} \;=\; \lambda\,(C - X)$ |
| jump | $\dot{C} \;=\; \frac{C-hX}{\sigma}\,(R - \pi - \rho) + h\,\lambda\,(C - X)$ |
| jump | $\dot{\pi} \;=\; \frac{1-(\phi/2)\pi^2}{1+(\phi/2)\pi^2}\,\left[\rho\,\pi - (\varepsilon/\phi)(MC - \frac{\varepsilon-1}{\varepsilon})\right]$ |
| algebraic | $Y \;=\; C\,/\,(1 - (\phi/2)\,\pi^2)$ |
| algebraic | $MC \;=\; Y^{\eta}\,(C - hX)^{\sigma}/A^{\eta+1}$ |
| algebraic | $R \;=\; \max(0,\;\rho + \phi_{\pi}\,\pi)$ |

One state, two jumps, three algebraic relations — the same saddle-path structure as the RBC, with the habit stock providing predetermined persistence and $Y$ tracking $C$ up to the adjustment-cost wedge.

# External habit – Steady state

- $\dot{X} = 0 \Rightarrow X^\star = C^\star$, so $C - hX = (1-h)\, C^\star$ in steady state.
- $\dot{\pi} = 0$ and the Taylor rule give $\pi^\star = 0$, $R^\star = \rho$.
- Substituting into $MC^\star = (\varepsilon - 1)/\varepsilon$:

$$C^{*\,\eta} \left((1-h)\, C^\star\right)^\sigma \; = \; (1-h)^\sigma \, C^{*\,\sigma+\eta} \; = \; \frac{\varepsilon - 1}{\varepsilon},$$

  hence

$$\boxed{C^\star \; = \; \left(\frac{\varepsilon - 1}{\varepsilon\,(1-h)^\sigma}\right)^{1/(\sigma+\eta)}}$$

- When $h = 0$ this collapses to the baseline steady state.

# Plan

# Internal habit – Hamiltonian

- Same felicity $u(C,X) = (C - hX)^{1-\sigma}/(1-\sigma)$, same habit law $\dot{X} = \lambda(C - X)$, but the household now **internalises** the effect of $C$ on $X$.
- The current-value Hamiltonian carries a costate $\mu$ on the habit stock in addition to the wealth costate $\lambda_B$:

$$\mathcal{H} \;=\; u(C,X) - \frac{N^{1+\eta}}{1+\eta} + \lambda_B \left[(R-\pi)\, B + w\, N - C\right] + \mu\, \lambda\, (C - X).$$

- The first-order conditions and the two costate equations give the full system. $C$ is no longer chosen as a forward-looking variable — it is pinned algebraically at every instant by the consumption FOC.

# Internal habit – First-order conditions

- $\partial\mathcal{H}/\partial C = 0$:

$$u_C - \lambda_B + \lambda\,\mu \;=\; 0 \quad\Longleftrightarrow\quad \boxed{\lambda_B \;=\; (C - hX)^{-\sigma} + \lambda\,\mu}$$

Compared with the no-habit case ($\lambda_B = u_C$), there is now an extra term $\lambda\,\mu$ capturing the marginal cost of raising the future habit stock.

- $\partial\mathcal{H}/\partial N = 0$: as before,

$$-N^{\eta} + \lambda_B\,w \;=\; 0 \quad\Longleftrightarrow\quad w \;=\; \frac{N^{\eta}}{\lambda_B}.$$

With $Y = C = A\,N$ and $A = 1$,

$$\boxed{MC \;=\; \frac{w}{A} \;=\; \frac{C^{\eta}}{\lambda_B}}$$

# Internal habit – Costate equations

- **Wealth costate** $\lambda_B$:

$$\boxed{\dot{\lambda}_B \;=\; \rho\,\lambda_B - \frac{\partial \mathcal{H}}{\partial B} \;=\; \lambda_B\,(\rho - R + \pi)}$$

- **Habit costate** $\mu$. With $u_X = -h\,(C - hX)^{-\sigma} = -h\,u_C$,

$$\dot{\mu} \;=\; \rho\,\mu - \frac{\partial \mathcal{H}}{\partial X} \;=\; \rho\,\mu - \big(u_X - \mu\,\lambda\big) \;=\; (\rho + \lambda)\,\mu - u_X.$$

  Substituting $u_X$:

$$\boxed{\dot{\mu} \;=\; (\rho + \lambda)\,\mu + h\,(C - hX)^{-\sigma}}$$

# Internal habit – Why C is algebraic

- The consumption FOC is an *algebraic* relation between $\lambda_B$, $C$, $X$ and $\mu$:

$$(C - hX)^{-\sigma} = \lambda_B - \lambda\,\mu.$$

- Solving for $C$ (the unknown not constrained by any time derivative in this rearrangement):

$$\boxed{C = h\,X + \left(\lambda_B - \lambda\,\mu\right)^{-1/\sigma}}$$

- So in the BVP the predetermined state is $X$; the forward-looking jumps are $\pi$, $\lambda_B$ and $\mu$; the algebraic variables are $C$, $R$ and $MC$.

- This reorganisation is essential for the IR's classification rule: $\lambda_B$ has its own ODE and so cannot also be the LHS of an algebraic equation. Writing $C$ as the LHS of the FOC keeps each endogenous on exactly one side of one equation.

# Internal habit – Full system

| role | equation |
|---|---|
| state | $\dot{X} \;=\; \lambda\,(C - X)$ |
| jump | $\dot{\lambda}_B \;=\; \lambda_B\,(\rho - R + \pi)$ |
| jump | $\dot{\mu} \;=\; (\rho + \lambda)\,\mu + h\,(C - hX)^{-\sigma}$ |
| jump | $\dot{\pi} \;=\; \frac{1-(\phi/2)\pi^2}{1+(\phi/2)\pi^2}\left[\rho\,\pi - (\varepsilon/\phi)(MC - \frac{\varepsilon-1}{\varepsilon})\right]$ |
| algebraic | $C \;=\; h\,X + (\lambda_B - \lambda\,\mu)^{-1/\sigma}$ |
| algebraic | $Y \;=\; C\,/\,(1 - (\phi/2)\,\pi^2)$ |
| algebraic | $MC \;=\; Y^{\eta}/(\lambda_B\,A^{\eta+1})$ |
| algebraic | $R \;=\; \max(0,\;\rho + \phi_\pi\,\pi)$ |

One state, three jumps, four algebraic equations. The terminal boundary anchors $\pi$, $\lambda_B$ and $\mu$ at the steady state; the initial state $X(0)$ is set by `initval(steady)`.

# Internal habit – Steady state, part 1

- $\dot{\pi} = 0$ and the Taylor rule give $\pi^\star = 0$ and $R^\star = \rho$.
- $\dot{X} = 0 \Rightarrow X^\star = C^\star$, so $C^\star - hX^\star = (1-h)\,C^\star$ in steady state.
- $\dot{\mu} = 0$:

$$(\rho + \lambda)\,\mu^\star + h\left((1-h)\,C^\star\right)^{-\sigma} \;=\; 0 \quad\Longleftrightarrow\quad \boxed{\mu^\star \;=\; -\,\frac{h\left((1-h)\,C^\star\right)^{-\sigma}}{\rho + \lambda} \;<\; 0}$$

The shadow value of the habit stock is *negative*: an extra unit of $X$ is a burden because it lowers future marginal utility.

# Internal habit – Steady state, part 2

- Consumption FOC at the steady state:

$$\lambda_B^\star \;=\; (C^\star - hX^\star)^{-\sigma} + \lambda\,\mu^\star \;=\; \big((1-h)\,C^\star\big)^{-\sigma}\left(1 - \frac{\lambda h}{\rho+\lambda}\right),$$

i.e.

$$\boxed{\lambda_B^\star \;=\; \big((1-h)\,C^\star\big)^{-\sigma}\,\frac{\rho+\lambda-\lambda h}{\rho+\lambda}}$$

- Substituting into $MC^\star = C^{*\,\eta}/\lambda_B^\star$ and equating to $(\varepsilon-1)/\varepsilon$:

$$\boxed{C^\star \;=\; \left(\frac{\varepsilon-1}{\varepsilon\,(1-h)^\sigma}\cdot\frac{\rho+\lambda-\lambda h}{\rho+\lambda}\right)^{1/(\sigma+\eta)}}$$

- At $h=0$ both correction factors vanish and $C^\star$ reduces to the baseline value $\big((\varepsilon-1)/\varepsilon\big)^{1/(\sigma+\eta)}$.

# Plan

# Experiment – The TFP boom

- All three variants share the same shock specification:

$$A(t) \;=\; 1 + 0.12 \cdot \mathbf{1}_{[0,3)}(t),$$

  i.e. productivity rises 12% over $[0,3)$ and snaps back to $1$ thereafter. The path is known to agents at $t=0$ (single belief, one segment).

- Each model is solved on a uniform Crank–Nicolson grid with `simulate(T=25, N=600)`. The economy starts at its (model-specific) steady state through `initval(steady)` — vacuous for the baseline since it has no state, and pinning $X(0) = X^{\star}$ for the habit variants.

- The boom drives marginal cost down (firms can produce more cheaply); the NKPC delivers deflation; the Taylor rule wants to cut $R$ below zero but is clamped at the ZLB. *What happens to consumption then depends sharply on preferences.*

# Experiment – Headline results

| scenario | $C$ on impact vs SS | $\pi$ trough | $R$ min | ZLB time |
|---|---|---|---|---|
| baseline (no habit) | $-4.7\%$ | $-6.60\%$ | $0$ | $9.8\%$ |
| external habit ($h = 0.7$) | $+7.0\%$ | $-5.00\%$ | $0$ | $8.0\%$ |
| internal habit ($h = 0.7$) | $+7.5\%$ | $-4.95\%$ | $0$ | $8.0\%$ |

- **Baseline**: deflation and a binding ZLB produce a real-rate-driven recession ($C$ drops 4.7% on impact) — the classic supply-side deflationary trap.
- **Habit variants**: the wealth effect from the productivity boom dominates the trap channel. $C$ jumps *up* 7% on impact and decays smoothly back; deflation is milder; the ZLB still touches zero briefly.
- "Trap mechanism" and "recession" are not synonymous: the ZLB binds in all three cases, but the consumption response is sign-flipped by habit smoothing.

# Experiment – Why the sign flip? Two competing channels

- A TFP boom activates **two opposing forces** on consumption:
  - *Real-rate (substitution) channel*: deflation + ZLB $\Rightarrow R - \pi > \rho$; the Euler $\dot{c}/c = (R - \pi - \rho)/\sigma$ pushes $c(0)$ *down* to be consistent with $c(\infty) = c^\star$.
  - *Wealth (income) channel*: higher $A$ raises lifetime resources, pushing $C(0)$ *up*.
- **Baseline** (log utility, no habit). Income and substitution effects of the rate cancel exactly on the *level* of $C$, so the Euler is the only pin-down. The trap delivers a high real rate, the substitution channel wins, and $C(0) < C^\star$.
- The next slide explains why habit *reverses* this outcome by amplifying wealth and damping substitution.

# Experiment – Why the sign flip? Habit amplifies wealth

- **Habit amplifies the wealth channel.** Active consumption $c = C - hX$ has SS level $(1-h)C^\star = 0.3\,C^\star$; the same $\sigma$ applied to a much smaller base means *sharper curvature* in $C$. The household becomes more eager to grab transitory windfalls before habit catches up.

- **Habit dampens substitution.** The state $X$ moves only at rate $\lambda = 0.5$, so a one-shot consumption surge is *not* immediately punished by a higher habit reference — the utility cost is spread over years.

- **Internal habit makes the wealth channel explicit.** The consumption FOC $C = hX + (\lambda_B - \lambda\mu)^{-1/\sigma}$ carries the forward-looking wealth costate $\lambda_B$, which integrates future productivity. The boom drops $\lambda_B$ on impact, and $C$ jumps up through the FOC — the textbook permanent-income response, absent from the baseline because $C$ there has no level offset and is tied directly to the Euler.

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